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Pure States of Simple C^*-algebras

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Pure states of simple C^*-algebras with identity are studied. We prove that pure states of such algebras have a product decomposition property, and that two pure states are unitarily equivalent if and only if they are asymptotically equal.

INTRODUCTION

In [4] Powers studied uniformly hyperfinite (UHF) C^*-algebras. He proved that factor states of such algebras can be characterized by a product decomposition property [4, Theorem 2.5], and he found necessary and sufficient conditions that two factor representations be quasiequivalent [4, Theorem 2.7]. Analogous results are also proved in [3]. In the present paper we shall derive the same type of results for pure states of simple C^*-algebras with identity, thus indicating how properties of UHF-algebras may be extended to general C^*-algebras.

A C^*-algebra $\mathfrak{A}$ is called a CCR-algebra if every irreducible representation of $\mathfrak{A}$ maps $\mathfrak{A}$ into the completely continuous operators. If a C^*-algebra $\mathfrak{A}$ has no nonzero CCR ideals, then we call $\mathfrak{A}$ an NGCR-algebra.

In [2, Lemma 4] Glimm proved that a separable NGCR-algebra with identity contains an ascending sequence of approximate matrix algebras of order $2, 4, \ldots, 2^n, \ldots$ with certain density properties, and we use these approximate matrix algebras to state our results.

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1. DEFINITIONS AND SIMPLE CONSEQUENCES

We use the notation and terminology developed by Glimm in [2]. We shall write 0_n for the $n$tuple $(0, \ldots, 0)$ and $[M]$ for the closed linear span of M, where M is a subset of a Hilbert space.

DEFINITION 1. Let $V(a_1, \ldots, a_n)$, $a_i \in \{0, 1\}$, and $B(n)$ be elements of a C^*-algebra, where n is a positive integer. We call

$$\{V(a_1, \ldots, a_n) V(b_1, \ldots, b_n)^*, B(n) : a_i, b_i \in \{0, 1\}\}$$

an *approximate matrix algebra* of order 2^n if the following axioms are satisfied:

(1) $V(a_1, \ldots, a_n)^* V(b_1, \ldots, b_n) = 0$ if $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$.

(2) $V(0_n) \geqslant 0$ and $\| V(a_1, \ldots, a_n)\| = 1$.

(3) $B(n) \geqslant 0$ and $\| B(n)\| = 1$.

(4) $V(a_1, \ldots, a_n)^* V(a_1, \ldots, a_n) B(n) = B(n)$.

DEFINITION 2. For each $n = 1, 2, \ldots$, let $V(a_1, \ldots, a_n)$, $a_i \in \{0, 1\}$, and $B(n)$ be elements of a C^*-algebra $\mathfrak{A}$. We call

$$\{V(a_1, \ldots, a_n) V(b_1, \ldots, b_n)^*, B(n) : a_i, b_i \in \{0, 1\} \text{ and } n = 1, 2, \ldots\}$$

an *approximate sequence of approximate matrix algebras* if the following properties are satisfied:

(1) We let $E(n) = \sum_{a_1, \ldots, a_n} V(a_1, \ldots, a_n) V(a_1, \ldots, a_n)^*$. For each $S \in \mathfrak{A}$ and each $\epsilon > 0$ there exist an n and a linear combination T of elements of the form $V(a_1, \ldots, a_n) V(b_1, \ldots, b_n)^*$ such that $\| E(n+1)(S - T) E(n+1)\| < \epsilon$.

(2) If $j \leqslant k$ and if $(a_1, \ldots, a_j) \neq (b_1, \ldots, b_j)$, then

$$V(a_1, \ldots, a_j)^* V(b_1, \ldots, b_k) = 0.$$

(3) If $k \geqslant 2$, then $V(a_1, \ldots, a_k) = V(a_1, \ldots, a_{k-1}) V(0_{k-1}, a_k)$.

(4) If $j < k$, then $V(a_1, \ldots, a_j)^* V(a_1, \ldots, a_j) V(0_{k-1}, a_k) = V(0_{k-1}, a_k)$.

(5) $V(0_n) \geqslant 0$ and $\| V(a_1, \ldots, a_n)\| = 1$.

(6) $V(a_1, \ldots, a_n)^* V(a_1, \ldots, a_n) B(n) = B(n)$.

(7) $\| B(n)\| = 1$ and $B(n) \geqslant 0$.

The difference between the axioms of Definition 2 and those in [2, Lemma 4] is so small that [2, Lemma 5] remains valid for an approximate sequence of approximate matrix algebras. This latter lemma therefore tells us about the matrix structure for such a sequence. The next three lemmas establish some properties of approximate sequences of approximate matrix algebras which we shall need later.

LEMMA 1. *Let*

$$\{V(a_1,\dots,a_n)\,V(b_1,\dots,b_n)^*,\ B(n) : a_i,\ b_i \in \{0,1\} \text{ and } n = 1,2,\dots\}$$

be an approximate sequence of approximate matrix algebras, and let $E(n)$ *be defined as in Definition 2. Then the following are true:*

(1) $\|E(n)\| = 1$ *and* $E(n) \geqslant 0$ *for* $n = 1, 2, \dots$.

(2) $V(a_1,\dots,a_n)\,V(b_1,\dots,b_n)^*\,E(n+1)$
$= \sum_{b=0,1} V(a_1,\dots,a_n,b)\,V(b_1,\dots,b_n,b)^*$.

(3) $E(n)\,E(m) = E(m)\,E(n) = E(m)$ *when* $n < m$.

(4) $V(a_1,\dots,a_n)\,V(b_1,\dots,b_n)^*$ *and* $E(p)$ *commute if* $n < p$.

(5) $V(i)\,V(j)^*\,V(p)\,V(k)^*\,E(n+1) = \delta_{j,p}V(i)\,V(k)^*\,E(n+1)$ *for all* $i, j, k, p \in \{0,1\}^n$. ($\delta_{j,j} = 1$ *and* $\delta_{j,p} = 0$ *if* $j \neq p$).

(6) $V(a_1,\dots,a_{n-1})\,V(b_1,\dots,b_{n-1})^*\,E(n+1)$
$= [V(a_1,\dots,a_{n-1},0)\,V(b_1,\dots,b_{n-1},0)^*$
$+ V(a_1,\dots,a_{n-1},1)\,V(b_1,\dots,b_{n-1},1)^*]\,E(n+1)$.

Proof. (1) Since $V(a_1,\dots,a_n)\,V(a_1,\dots,a_n)^* \geqslant 0$ for all $(a_1,\dots,a_n) \in \{0,1\}^n$, we have $E(n) \geqslant 0$.

$$V(b_1,\dots,b_n)^*\,[V(b_1,\dots,b_n)\,V(a_1,\dots,a_n)^*]\,V(a_1,\dots,a_n)\,B(n) = B(n)$$

is a consequence of Definition 2, Axiom (6). Since all the $V(a_1,\dots,a_n)$ and $B(n)$ have norm one, we get from the Cauchy–Schwarz inequality that $\|V(b_1,\dots,b_n)\,V(a_1,\dots,a_n)^*\| = 1$ for $(a_1,\dots,a_n), (b_1,\dots,b_n) \in \{0,1\}^n$. This together with the fact that $V(a_1,\dots,a_n)^*\,V(b_1,\dots,b_n) = 0$ if $(a_1,\dots,a_n) \neq (b_1,\dots,b_n)$, implies that $\|E(n)\| = 1$.

(2) In the following we use without comment Definition 2, Axioms 2, 3, and 4.

$$V(a_1,\dots,a_n)\,V(b_1,\dots,b_n)^*\,V(c_1,\dots,c_{n+1})\,V(c_1,\dots,c_{n+1})^* = 0 \quad \text{if } (b_1,\dots,b_n) \neq (c_1,\dots,c_n)$$

and

$$\begin{aligned} &V(a_1,\dots,a_n)\,V(b_1,\dots,b_n)^*\,V(b_1,\dots,b_n,c_{n+1})\,V(b_1,\dots,b_n,c_{n+1})^* \\ &= V(a_1,\dots,a_n)\,V(b_1,\dots,b_n)^*\,V(b_1,\dots,b_n)\,V(0_n,c_{n+1})\,V(b_1,\dots,b_n,c_{n+1})^* \\ &= V(a_1,\dots,a_n)\,V(0_n,c_{n+1})\,V(b_1,\dots,b_n,c_{n+1})^* \\ &= V(a_1,\dots,a_n,c_{n+1})\,V(b_1,\dots,b_n,c_{n+1})^*. \end{aligned}$$

From these equalities we can easily prove (2).

(3) From (2) we get $E(n)\,E(n+1) = E(n+1)$. Since $E(n)$ is selfadjoint for each n, it follows that $E(n+1)\,E(n) = E(n+1)$. We suppose $k > n$ and get

$$\begin{aligned} E(n)\,E(k) &= E(n)\,E(n+1) \cdots E(k-1)\,E(k) = E(k) \\ &= E(k)\,E(k-1) \cdots E(n+1)\,E(n) = E(k)\,E(n). \end{aligned}$$

(4) We prove the assertion by induction with respect to the difference $p - n$. We suppose first that $p - n = 1$. From (2) and $E(n) = E(n)^*$ it follows that

$$\begin{aligned} &E(n+1)\,V(a_1,\dots,a_n)\,V(c_1,\dots,c_n)^* \\ &\quad = [V(c_1,\dots,c_n)\,V(a_1,\dots,a_n)^*\,E(n+1)]^* \\ &\quad = \Big[\sum_{b=0,1} V(c_1,\dots,c_n,b)\,V(a_1,\dots,a_n,b)^*\Big]^* \\ &\quad = \sum_{b=0,1} V(a_1,\dots,a_n,b)\,V(c_1,\dots,c_n,b)^* \\ &\quad = V(a_1,\dots,a_n)\,V(c_1,\dots,c_n)^*\,E(n+1). \end{aligned}$$

We suppose that the assertion is true for $p - n = s \geqslant 1$ and that $p - n = s + 1$. From (2) and (3) we get

$$\begin{aligned} V(a_1,\dots,a_n)\,V(c_1,\dots,c_n)^*\,E(p) &= V(a_1,\dots,a_n)\,V(c_1,\dots,c_n)^*\,E(n+1)\,E(p) \\ &= \sum_{b=0,1} V(a_1,\dots,a_n,b)\,V(c_1,\dots,c_n,b)^*\,E(p) \\ &= E(p) \sum_{b=0,1} V(a_1,\dots,a_n,b)\,V(c_1,\dots,c_n,b)^* \\ &= E(p)\,V(a_1,\dots,a_n)\,V(c_1,\dots,c_n)^*\,E(n+1) \\ &= E(p)\,E(n+1)\,V(a_1,\dots,a_n)\,V(c_1,\dots,c_n)^* \\ &= E(p)\,V(a_1,\dots,a_n)\,V(c_1,\dots,c_n)^*. \end{aligned}$$

(5) and (6) are proved in the same way as is [2, Lemma 5].

By a simple induction argument the next lemma follows from Lemma 1.

LEMMA 2. *Let*

$$\{V(a_1,\dots,a_n)\,V(b_1,\dots,b_n)^*,\ B(n) : a_i, b_i \in \{0,1\} \text{ and } n = 1, 2,\dots\}$$

be an approximate sequence of approximate matrix algebras in a C^*-algebra $\mathfrak{A}$. For each n we let $\mathfrak{B}_n$ be the $*$-algebra generated by all $V(a_1, \dots, a_m) V(c_1, \dots, c_m)^*$ such that $0 < m \leqslant n$ and $(a_1, \dots, a_m)$, $(c_1, \dots, c_m) \in \{0, 1\}^m$.

Then for each $x \in \mathfrak{B}_n$ there exist complex numbers $a_{(a_1, \dots, a_n), (c_1, \dots, c_n)}$ such that

$$xE(n+1) = \sum_{\substack{(a_1, \dots, a_n) \\ (c_1, \dots, c_n)}} a_{(a_1, \dots, a_n), (c_1, \dots, c_n)} V(a_1, \dots, a_n) V(c_1, \dots, c_n)^* E(n+1).$$

We illustrate the proof by an example. We let

$$x = V(1, 1) V(0, 0)^* V(0, 0, 1) V(1, 1, 1)^*,$$

and it follows that

$$\begin{aligned} xE(4) &= V(1, 1) V(0, 0)^* E(3) E(4) V(0, 0, 1) V(1, 1, 1)^* \\ &= [V(1, 1, 0) V(0, 0, 0)^* + V(1, 1, 1) V(0, 0, 1)^*] \\ &\quad \times V(0, 0, 1) V(1, 1, 1)^* E(4) \\ &= V(1, 1, 1) V(1, 1, 1)^* E(4). \end{aligned}$$

LEMMA 3.

$$\{V(a_1, \dots, a_n) V(b_1, \dots, b_n)^*, B(n) : a_i, b_i \in \{0, 1\} \text{ and } n = 1, 2, \dots\}$$

and $\mathfrak{B}_n$ are defined in Lemma 2.

Then for each $y \in \mathfrak{A}$ we have

$$\begin{aligned} z &= E(n+1) \\ &\quad \times \left[\sum_{(a_1, \dots, a_n)} V(a_1, \dots, a_n) V(0_n)^* y V(0_n) V(a_1, \dots, a_n)^* \right] E(n+1) \in \mathfrak{B}_n{}^c \end{aligned}$$

where $\mathfrak{B}_n{}^c$ is the commutant to $\mathfrak{B}_n$ in $\mathfrak{A}$.

Proof. In this proof we use without comment the axioms of Definition 2 and the results in Lemma 1. We have for $j, k \in \{0, 1\}^n$

$$\begin{aligned} V(j) V(k)^* z &= E(n+1) V(j) V(k)^* V(k) V(0_n)^* y V(0_n) V(k)^* E(n+1) \\ &= V(j) V(k)^* V(k) V(0_n)^* E(n+1) y V(0_n) V(k)^* E(n+1) \\ &= V(j) V(0_n)^* E(n+1) y V(0_n) V(k)^* E(n+1) \\ &= E(n+1) V(j) V(0_n)^* y V(0_n) V(j)^* V(j) V(k)^* E(n+1) \\ &= E(n+1) V(j) V(0_n)^* y V(0_n) V(j)^* E(n+1) V(j) V(k)^* \\ &= z V(j) V(k)^*. \end{aligned}$$

We let $x \in \mathfrak{B}_n$. By Lemma 2 there exist complex numbers $a_{i,j}$, $i, j \in \{0, 1\}^n$, such that

$$xE(n+1) = \sum_{i,j \in \{0,1\}^n} a_{i,j} V(i)\, V(j)^*\, E(n+1).$$

This implies that

$$xz = \sum_{i,j} a_{i,j} V(i)\, V(j)^*\, z \quad \text{and} \quad zx = \sum_{i,j} a_{i,j} z V(i)\, V(j)^*.$$

It follows now that $xz = zx$, and we have $z \in \mathfrak{B}_n{}^c$.

2. Two Variations of Glimm's Lemma

We need two small variations on the fundamental [2, Lemma 4].

LEMMA 4. *Let $\mathfrak{A}$ be a simple, separable* NGCR-*algebra with identity, and let f be a pure state. Then $\mathfrak{A}$ contains an approximate sequence of approximate matrix algebras such that $f(B(n)) = 1$ for all n.*

Proof. We let $S_0, S_1, \ldots$ be a dense subset of the selfadjoint elements in $\mathfrak{A}$. We change the proof of [2, Lemma 4] such that we in addition get $f(B(n)) = 1$ for all n. The induction step in the proof need be changed in only two places.

First, in the seventh line from the top of [2, p. 577], we let $\mu = f$. This is possible since $f(B(n)) = 1$.

The other change is in lines 11–13 of page 578. There we let $\varphi = \varphi_f$ and $y = x_f$. This is possible since $\varphi_f(B_o)$ is noncompact, because $\mathfrak{A}$ is simple, and since $\varphi_f(B_o)\, x_f = x_f$ (line 10, p. 578).

From the 13th line from the bottom of page 579 in Glimm's proof it follows that $\varphi_f(B(n+1))\, x_f = x_f$. This implies that $f(B(n+1)) = 1$.

We have now found elements $V(a_1, \ldots, a_n)$ and $B(n)$ such that the axioms (2)–(7) in Definition 2 are satisfied and elements $T_n \in \mathfrak{M}(n)$ ($\mathfrak{M}(n)$ is the linear span of elements of the form $V(a_1,\ldots,a_n) V(b_1,\ldots,b_n)^*$) such that $\| E(n+1)(S_n - T_n)\, E(n+1)\| < 1/n$.

We let $\epsilon > 0$ and $S \in \mathfrak{A}$ be arbitrary. There exist selfadjoint elements S' and S'' such that $S = S' + iS''$. We chose k_1 and k_2 such that $\| S' - S_{k_1} \| < \epsilon/4$, $\| S'' - S_{k_2} \| < \epsilon/4$, $1/k_1 < \epsilon/4$ and $1/k_2 < \epsilon/4$. Since $\| E(n)\| = 1$ and $E(n)\, E(m) = E(m)$ if $n < m$, it follows by an $\epsilon/4$-argument that

$$\| E(p+1)[S - (T_{k_1} + iT_{k_2})]\, E(p+1)\| < \epsilon,$$

where $p = \max(k_1, k_2)$. By Lemma 2 there is a $T \in \mathfrak{M}(p)$ such that $(T_{k_1} + iT_{k_2}) E(p+1) = TE(p+1)$. This implies that $\| E(p+1)(S - T) E(p+1)\| < \epsilon$, and we are done.

LEMMA 5. *Let $\mathfrak{U}$ be a simple* NGCR-*algebra with identity. Let f_1 and f_2 be two pure states such that f_1 and f_2 are not unitary equivalent. Let*

$$\{V(a,\ldots, a_n)\, V(b_1,\ldots, b_n)^*, B(n) : a_i, b_i \in \{0, 1\}\}$$

be an approximate matrix algebra such that $f_1(B(n)) = 1$. Then there exists an approximate matrix algebra

$$\{V(a_1,\ldots, a_{n+1})\, V(b_1,\ldots, b_{n+1})^*, B(n+1) : a_i, b_i \in \{0, 1\}\}$$

such that $f_1(B(n+1)) = 1$ and $f_2(E(n+1)) = 0$, where

$$E(n+1) = \sum_{(a_1,\ldots,a_{n+1})} V(a_1,\ldots, a_{n+1})\, V(a_1,\ldots, a_{n+1})^*,$$

and such that

$$(1) \quad V(a_1,\ldots, a_{n+1}) = V(a_1,\ldots, a_n)\, V(0_n, a_{n+1})$$

and

$$(2) \quad V(a_1,\ldots, a_n)^*\, V(a_1,\ldots, a_n)\, V(0_n, a_{n+1}) = V(0_n, a_{n+1}).$$

Proof. The proof is analogous to the proof of the induction step in [2, Lemma 4]. We make some small changes.

We let φ_i and x_i respectively be the induced representation and induced vector of f_i. We let H_i be the Hilbert space on which φ_i acts. The elements D_0, D_1, B_σ, $B_{2\sigma}$, and V, which we mention in the following proof, are defined on page 578 in Glimm's proof, and the function f_σ is defined on page 577.

First, in the seventh line from the top of page 577 we let $\mu = f_1$. This is possible since $f_1(B(n)) = 1$.

In lines 10–18 on page 578 we make the following changes. We let $\varphi = \varphi_1$ (line 11). This is possible since $\varphi_1(B_\sigma)\, x_1 = x_1$ and $\mathfrak{U}$ is simple, hence $\varphi_1(B_\sigma)$ is noncompact. We let $y = x_1$. This is possible since $\varphi_1(B_\sigma)\, x_1 = x_1$, which implies that $x_1 \in \text{Range } \varphi_1(B_\sigma)$.

We define N by

$$N = [\varphi_2(V(i)^*)\, x_2 : i \in \{0, 1\}^n], \tag{2.1}$$

which is a finite dimensional subspace of H_2. We require in addition of C_0 and U in the lines 14 and 17 that

$$\varphi_2(C_0)(B_{2\sigma} N) = \{0\} \tag{2.2}$$

and that

$$\varphi_2(U^*)(f_\sigma(D_1)N) \subset N. \tag{2.3}$$

This is possible by an application of [1, Theorem 2.8.3], since $\dim[f_\sigma(D_1)N] \leqslant \dim N < \infty$, and since f_1 and f_2 are not unitarily equivalent.

By making these changes in the induction step of Glimm's proof we find an approximate matrix algebra

$$\{V(a_1, ..., a_{n+1})\ V(b_1, ..., b_{n+1})^*, B(n+1) : a_i\,, b_i \in \{0, 1\}\}$$

such that (1) and (2) are satisfied. It remains to prove that our changes imply that $f_1(B(n+1)) = 1$ and $f_2(E(n+1)) = 0$.

By (2.2) we have $\varphi_2(D_0)(N) = \{0\}$, and hence $\varphi_2(V)(N) = \{0\}$. Since $V^* = f_\sigma(D_0)\ U^* f_\sigma(D_1)$, by (2.3) we have $\varphi_2(V^*)(N) = \{0\}$. From the definition of $V(0_n\,, 1)$ and $V(0_{n+1})$ we get $\varphi_2(V(0_n\,, 1)^*)(N) = \{0\}$ and $\varphi_2(V(0_{n+1}))(N) = \{0\}$. (2.2) implies now that

$$\varphi_2(V(0_n\,, 1)^*\ V(a_1, ..., a_n)^*\ x_2 = 0$$

and

$$\varphi_2(V(0_{n+1})^*\ V(a_1, ..., a_n)^*\ x_2 = 0 \qquad \text{for all} \quad (a_1, ..., a_n) \in \{0, 1\}^n.$$

This implies that $\varphi_2(E(n+1))\, x_2 = 0$, and hence $f_2(E(n+1)) = 0$.

From line 13 from the bottom of page 579 we get $\varphi(B(n+1))\, y = y$. Since we have chosen $\varphi = \varphi_1$ and $y = x_1$, we then get $\varphi_1(B(n+1))\, x_1 = x_1$ and hence $f_1(B(n+1)) = 1$.

We suppose we have two approximate matrix algebras which satisfy (1) and (2) in Lemma 5. Then, in the same way as in the proof of [2, Lemma 5], we can show the following: $\mathfrak{M}(n)$ is the set of all finite linear combinations of elements of the form $V(a_1, ..., a_n)\ V(b_1, ..., b_n)^*$. For each representation φ of $\mathfrak{A}$,

$$\varphi(\mathfrak{M}(n)) \mid_{[\text{range}\varphi(E(n+1))H_\varphi]}$$

is a $2^n \times 2^n$ matrix algebra with matrix units

$$\varphi(V(a_1, ..., a_n)\ V(b_1, ..., b_n)^* \mid_{[\text{range}\varphi(E(n+1))H_\varphi]} .$$

This justifies Definition 1 of an approximate matrix algebra.

3. Main Results

We prove in Theorem 1 that pure states of a simple separable C^*-algebra with identity have a product decomposition property. Moreover, we prove in Theorem 2 that two pure states of a simple

C^*-algebra with identity are unitarily equivalent if and only if they are asymptotically equal. The following result is well-known, and is stated without proof.

LEMMA 6. *Let* $\mathfrak{U}$ *be a simple* C^**-algebra with identity. Then either* $\mathfrak{U}$ *is an* NGCR*-algebra or else* $\mathfrak{U}$ *is *-isomorphic with an* $n \times n$ *matrix algebra, where* n *is finite.*

THEOREM 1. *Let* $\mathfrak{U}$ *be a simple separable* C^**-algebra with identity. We suppose that* $\mathfrak{U}$ *is not *-isomorphic with any* $n \times n$ *matrix algebra such that* n *is finite. Let* f *be a pure state of* $\mathfrak{U}$.

Then $\mathfrak{U}$ *contains an approximate sequence of approximate matrix algebras*

$$\{V(a_1, ..., a_n)\ V(b_1, ..., b_n)^*,\ B(n) : a_i, b_i \in \{0, 1\}^n \text{ and } n = 1, 2, ...\}$$

such that the following are satisfied:

We let $\mathfrak{A}$ *be the* C^**-algebra generated by* $\{V(a_1, ..., a_n)\ V(b_1, ..., b_n)^*: a_i, b_i \in \{0, 1\}$ *and* $n = 1, 2, ...\}$, *and we let* $\mathfrak{M}(n)$ *be the set of all linear combinations of* $V(a_1, ..., a_n)\ V(b_1, ..., b_n)^*$. *Then for each* $\epsilon > 0$ *and each* $x \in \mathfrak{A}$, *there is an* n *such that*

$$|f(xy) - f(x) f(y)| < \epsilon \| y \| \qquad \text{for} \quad y \in \mathfrak{M}(n)^c.$$

($\mathfrak{M}(n)^c$ *is the commutant of* $\mathfrak{M}(n)$ *in* $\mathfrak{U}$.)

Proof. In this proof we use the axioms of Definition 2 and Lemma 1 without comment.

By Lemma 6 $\mathfrak{U}$ is an NGCR algebra. We use Lemma 4 and choose an approximate sequence of approximate matrix algebras such that $f(B(n)) = 1$ for all n.

$$\begin{aligned} E(n)\, B(n) &= E(n)\, V(0_n)\, V(0_n)\, B(n) \\ &= \sum_{(a_1, \dots, a_n)} V(a_1, ..., a_n)\, V(a_1, ..., a_n)^*\, V(0_n)\, V(0_n)\, B(n) \\ &= V(0_n)\, V(0_n)\, V(0_n)\, V(0_n)\, B(n) \\ &= B(n). \end{aligned}$$

Since $f(B(n)) = 1$ and $\| B(n)\| = 1$, we have

$$(\varphi_f(B(n))\, x_f, x_f) = 1 = \| \varphi_f(B(n))\, x_f \| \cdot \| x_f \|.$$

Thus $\varphi_f(B(n))\, x_f$ is proportional to x_f, and so is equal to x_f. Since $E(n)\, B(n) = B(n)$, we have

$$\varphi_f(E(n))\, x_f = x_f \quad \text{and} \quad f(E(n)) = 1 \qquad \text{for} \quad n = 1, 2, 3\ldots. \tag{3.1}$$

We have now to prove the following assertion:

$$f\,|_{\mathfrak{A}} \text{ is a pure state}.$$

We prove first that $f\,|_{\mathfrak{A}}$ has a unique extension to $\mathfrak{U}$. Suppose then that g is a state such that $f\,|_{\mathfrak{A}} = g\,|_{\mathfrak{A}}$. In the same way as we prove $\varphi_f(B(n))\, x_f = x_f$, we prove that $\varphi_g(E(n))\, x_g = x_g$ for $n = 1, 2,\ldots$. From this and (3.1) we get

$$f(\cdot) = f(E(n) \cdot E(n)) \quad \text{and} \quad g(\cdot) = g(E(n) \cdot E(n)) \qquad \text{for} \quad n = 1, 2,\ldots \tag{3.2}$$

We let $S \in \mathfrak{U}$ and $\epsilon > 0$ be arbitrary and choose n and $T \in \mathfrak{A}$ such that $\| E(n)(T - S)\, E(n)\| < \epsilon$. By (3.2) it follows that

$$\begin{aligned} |f(S) - g(S)| &= |f(T) - g(T) + f(S - T) - g(S - T)| \\ &= |(f - g)(E(n)(S - T)\, E(n))| < 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have $f(S) = g(S)$. Next we prove that $f\,|_{\mathfrak{A}}$ is pure. We suppose $f\,|_{\mathfrak{A}} = \frac{1}{2}(h + g)$, where h and g are states of $\mathfrak{A}$. We extend h and g to $\mathfrak{U}$ and call the extensions h' and g'. Since we have just proved that $f\,|_{\mathfrak{A}}$ has a unique extension to $\mathfrak{U}$, it follows that $f = \frac{1}{2}(h' + g')$. f is pure, hence $f = h' = g'$, and we have proved the assertion.

We let $\mathfrak{B}_n$ be the $*$-algebra generated by $\{V(a_1,\ldots, a_k)\, V(b_1,\ldots, b_k)^*$: $a_i, b_i \in \{0, 1\}, k \leqslant n\}$. Since $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{B}_n}^{\text{norm}}$, it is sufficient to prove the theorem for each $x \in \bigcup_{n=1}^{\infty} \mathfrak{B}_n$.

We let $x \in \mathscr{B}_n$ and $\epsilon > 0$ be given. We choose $\delta > 0$ such that

$$\| x \| \cdot \delta + \delta\, |f(x)| + \delta(1 + \delta) < \epsilon.$$

$\{\mathfrak{B}_n\}_{n=1}^{\infty}$ is an ascending sequence of $*$-algebras such that $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{B}_n}^{\text{norm}}$, and $f\,|_{\mathfrak{A}}$ is a pure state, in particular a factor state. We copy the proof of [4, Theorem 2.5(i)–(ii)] and find $m > n$ such that

$$|f(xy) - f(x)\, f(y)| \leqslant \delta\, \| y \| \qquad \text{for all} \quad y \in \mathfrak{B}_m{}^c \cap \mathfrak{A}. \tag{3.3}$$

We let $y \in \mathfrak{M}(m)^c$, and we suppose without loss of generality that $\| y \| = 1$. We need now the following assertion:

For each $\delta > 0$ and each $S \in \mathfrak{U}$ there exist k and $T \in \mathfrak{B}_k$ such that $\| T \| \leqslant \| S \| + \delta$ and $\| E(k+1)(S - T) E(k+1) \| < \delta$.

We choose p and T' such that $\| E(p+1)(S - T') E(p+1) \| < \delta$. We define $k = p + 1$ and $T = E(p+1) T' E(p+1)$.

$$\| T \| = \| E(p+1) T'E(p+1) \|$$
$$\leqslant \| E(p+1)(S - T') E(p+1) \| + \| E(p+1) SE(p+1) \| < \delta + \| S \|,$$

since $\| E(p+1) \| = 1$. We get

$$\| E(p+2)(S - T) E(p+2) \|$$
$$= \| E(p+2)(E(p+1) SE(p+1) - E(p+1) T'E(p+1)) E(p+2) \|$$
$$\leqslant \| E(p+2) \| \cdot \| E(p+1)(S - T') E(p+1) \| \cdot \| E(p+2) \| < \delta$$

and we have proved the assertion.

By the assertion we can find $k > \max(m, n)$ and $z \in \mathfrak{B}_k$ such that

$$\| z \| < 1 + \delta \qquad \text{and} \qquad \| E(k+1)(z - y) E(k+1) \| < \delta/2^m. \tag{3.4}$$

Since $\| V(a_1, ..., a_m) V(0_m)^* \| = 1$, we have by (3.4)

$$\| V(a_1, ..., a_m) V(0_m)^* E(k+1)(z - y) E(k+1) V(0_m) V(a_1, ..., a_m)^* \|$$
$$< \delta/2^m \qquad \text{for all} \quad (a_1, ..., a_m) \in \{0, 1\}^m. \tag{3.5}$$

$$\sum_{(a_1, ..., a_m)} V(a_1, ..., a_m) V(0_m)^* E(k+1)(y - z) E(k+1) V(0_m) V(a_1, ..., a_m)^*$$
$$= \sum_{(a_1, ..., a_m)} E(k+1) y \, V(a_1, ..., a_m) V(0_m)^* V(0_m)$$
$$\times V(a_1, ..., a_m)^* E(k+1) - E(k+1)$$
$$\times \left(E(m+1) \sum_{(a_1, ..., a_m)} V(a_1, ..., a_m) V(0_m)^* z V(0_m) V(a_1, ..., a_m)^* E(m+1) \right)$$
$$\times E(k+1)$$
$$= E(k+1) y \, E(m) E(k+1) - E(k+1) z' E(k+1)$$
$$= E(k+1)(y - z') E(k+1),$$

where z' is defined by

$$z' = E(m+1) \sum_{(a_1, ..., a_m)} V(a_1, ..., a_m) V(0_m)^* z V(0_m) V(a_1, ..., a_m)^* E(m+1) \tag{3.6}$$

We add the inequalities in (3.5) and get

$$\| E(k+1)(y - z') E(k+1) \| < \delta. \tag{3.7}$$

From (3.4) it follows that

$$\| V(a_1, \ldots, a_m) V(0_m)^* z V(0_m) V(a_1, \ldots, a_m)^* \| < 1 + \delta.$$

Since

$$V(a_1, \ldots, a_m)^* V(b_1, \ldots, b_m) = 0 \qquad \text{if} \quad (a_1, \ldots, a_m) \neq (b_1, \ldots, b_m),$$

we get

$$\Big\| \sum_{(a_1, \ldots, a_m)} V(a_1, \ldots, a_m) V(0_m)^* z V(0_m) V(a_1, \ldots, a_m)^* \Big\| < 1 + \delta.$$

By (3.6) this gives

$$\| z' \| < 1 + \delta. \tag{3.8}$$

By Lemma 3 we have $z' \in \mathfrak{B}_m{}^c \cap \mathfrak{A}$. This implies by (3.3) and (3.8) that

$$| f(xz') - f(x) f(z') | < \delta \| z' \| \leqslant \delta(1 + \delta). \tag{3.9}$$

Since $x \in \mathfrak{B}_n$, $z' \in \mathfrak{B}_{m+1}$ and $k + 1 > \max(k, n)$, we have by (3.2) and (3.7) that

$$| f(xz') - f(xy) | \leqslant \| x \| \delta, \tag{3.10}$$

because

$$\begin{aligned} &| f(xz') - f(xy) | \\ &\quad = | f(xE(k+1) z'E(k+1)) - f(xE(k+1) yE(k+1)) | \\ &\quad \leqslant \| xE(k+1) z'E(k+1) - xE(k+1) yE(k+1) \| \leqslant \| x \| \cdot \delta. \end{aligned}$$

Moreover, we have by (3.2) and (3.7) that

$$| f(z') - f(y) | = | f(E(k+1)(z' - y) E(k+1)) | \leqslant \delta. \tag{3.11}$$

(3.9), (3.10), and (3.11) imply

$$| f(xy) - f(x) f(y) | \leqslant \| x \| \cdot \delta + \delta \cdot | f(x) | + \delta(1 + \delta) < \epsilon,$$

and we are done.

THEOREM 2. *Let* $\mathfrak{A}$ *be a simple* C^**-algebra with identity. We suppose that* $\mathfrak{A}$ *is not* $*$*-isomorphic with any* $n \times n$ *matrix algebra* *such that n is finite. Let f_1 and f_2 be two pure states of $\mathfrak{U}$. Then the following are equivalent:*

(1) *f_1 and f_2 are unitarily equivalent.*

(2) *There is an approximate matrix algebra*

$$\{V(a_1)\, V(b_1)^*, B(1) : a_1, b_1 \in \{0, 1\}\}$$

such that

$$f_1(B(1)) = 1 \qquad \textit{and} \qquad \|(f_1 - f_2)\,|_{\mathfrak{M}(1)^c}\| = 0.$$

(3) *There is an approximate matrix algebra*

$$\{V(a_1, ..., a_n)\, V(b_1, ..., b_n)^*, B(n) : a_i, b_i \in \{0, 1\}\}$$

such that

$$f_1(B(n)) = 1 \qquad \textit{and} \qquad \|(f_1 - f_2)\,|_{\mathfrak{M}(n)^c}\| < 1.$$

$\mathfrak{M}(n)$ is the linear span of the elements $V(a_1, ..., a_n)\, V(b_1, ..., b_n)^*$, and $\mathfrak{M}(n)^c$ is the commutant of $\mathfrak{M}(n)$ in $\mathfrak{U}$.

Proof. By Lemma 6, $\mathfrak{U}$ is a simple NGCR-algebra with identity. (1) $\rightarrow$ (2): We suppose $f_1 \sim f_2$. We define $\pi = \pi_{f_1}$. If π is a one-dimensional representation, the theorem is trivially satisfied. We suppose that π is at least two-dimensional, that $f_1(\cdot) = (\pi(\cdot)\, x_1, x_1)$, that $f_2(\cdot) = (\pi(\cdot)\, x_2, x_2)$, and that $x_2 = \lambda x_1 + \mu z$ where $x_1 \perp z$, $\| z \| = 1$ and $\lambda, \mu \in \mathbb{C}$. By [1, Theorem 2.8.3] there exist elements D and U of $\mathfrak{U}$ such that $D \geqslant 0$, $\| D \| = 1$, $\pi(D)\, x_1 = x_1$, $\pi(D) z = 0$, U is unitary, and $\pi(U)\, x_1 = z$.

For each $\epsilon > 0$ in $(0, 1)$ we let f_ϵ be the function defined by: $f_\epsilon((-\infty, 1 - \epsilon]) = 0$, $f_\epsilon([1 - (\epsilon/2), \infty)) = 1$, and f_ϵ is linear on $[1 - \epsilon, 1 - (\epsilon/2)]$. We define

$$V = f_{1/2}(I - D)\, U f_{1/2}(D).$$

We prove now that $f_{1/2}(I - D) f_{1/2}(D) = 0$. We define g by $g(t) = f_{1/2}(1 - t) f_{1/2}(t)$. Since $f_{1/2} = 0$ on $[0, \frac{1}{2}]$ and $\operatorname{sp}(D) \subset [0, 1]$, it follows that $g = 0$ on $\operatorname{sp}(D)$. This implies $g(D) = 0$. Since $f_{1/2}(I - D) f_{1/2}(D) = 0$, it follows that $V^2 = 0$. We have

$$\pi(V)\, x_1 = z \qquad \text{and} \qquad \pi(V^*) z = x_1. \tag{13.12}$$

We define

$$V(1) = V k(V^* V), \qquad \text{where} \quad k(t) = (f_{1/2}(t)\, t^{-1})^{1/2}, \qquad k(0) = 0,$$
$$V(0) = f_{1/2}(V^* V),$$

and

$$B(1) = f_{1/4}(V^* V).$$

Next we want to prove that

$$\{V(i)\, V(j)^*, B(1) : i, j \in \{0, 1\}\}$$

is an approximate matrix algebra. $V(1)^*\, V(0) = 0$, since $(V^*)^2 = 0$. Moreover, $V(0)^*\, V(1) = 0$, since $V^2 = 0$. This means that Axiom (1) in Definition 1 is satisfied. Axioms (2) and (3) are trivially satisfied. Since

$$V(1)^*\, V(1) = k(V^*V)\, V^*Vk(V^*V) = f_{1/2}(V^*V),$$

it follows that $V(1)^*\, V(1)\, B(1) = B(1)$, because $f_{1/2}f_{1/4} = f_{1/4}$. Since $f_{1/2}f_{1/4} = f_{1/4}$, it follows that $V(0)^*\, V(0)\, B(1) = B(1)$, and Axiom 4 is satisfied. Thus we have proved that

$$\{V(i)\, V(j)^*, B(1) : i, j \in \{0, 1\}\}$$

is an approximate matrix algebra.

We define G by

$$G = \lambda V(0)\, V(0)^* + \mu V(1)\, V(0)^*.$$

From (13.12) we get

$$\pi(V(0)\, V(0)^*)\, x_1 = \pi([f_{1/2}(V^*V)]^2)\, x_1 = x_1\,,$$
$$\pi(V(1)\, V(0)^*)\, x_1 = \pi(Vk(V^*V)\, f_{1/2}(V^*V))\, x_1 = z,$$
$$\pi(V(0)\, V(1)^*)z = \pi(f_{1/2}(V^*V)\, k(V^*V)\, V^*)z = x_1\,,$$

and hence

$$\pi(G)\, x_1 = \lambda x_1 + \mu z = x_2\,.$$

We get

$$\pi(V(0)\, V(0)^*)z = \pi(V(0)\, V(0)^*\, V(1)\, V(0)^*)\, x_1 = 0$$

and

$$\pi(V(0)\, V(1)^*)\, x_1 = \pi(V(0)\, V(1)^*\, V(0)\, V(1)^*)z = 0.$$

This implies

$$\begin{aligned}\pi(G^*)(\lambda x_1 + \mu z) &= (\bar{\lambda} V(0)\, V(0)^* + \bar{\mu} V(0)\, V(1)^*)(\lambda x_1 + \mu z)\\ &= (|\lambda|^2 + |\mu|^2)\, x_1 = 1 \cdot x_1 = x_1\,.\end{aligned}$$

We get

$$\pi(G^*G)\, x_1 = x_1\,.$$

We let $A \in \mathfrak{M}(1)^c$.

We get

$$f_2(A) = (\pi(A)\, x_2\,,\, x_2) = (\pi(A)\,\pi(G)\, x_1\,,\, \pi(G)\, x_1)$$
$$= f_1(G^*AG) = f_1(AG^*G) = f_1(A)$$

since G and A commute and $\pi(G^*G)\, x_1 = x_1$.

(2) $\rightarrow$ (3) is trivial.

(3) $\rightarrow$ (1): We suppose $f_1 \not\sim f_2$, and we let

$$\{V(a_1\,,\ldots, a_n)\; V(b_1\,,\ldots, b_n)^*,\, B(n) : a_i\,,\, b_i \in \{0, 1\}\}$$

be an approximate matrix algebra such that $f_1(B(n)) = 1$. By Lemma 5 we choose an approximate matrix algebra

$$\{V(a_1\,,\ldots, a_{n+1})\; V(b_1\,,\ldots, b_{n+1})^*,\, B(n+1) : a_i\,,\, b_i \in \{0, 1\}\}$$

such that (1) and (2) in Lemma 5 are satisfied and such that $f_1(B(n+1)) = 1$ and $f_2(E(n+1)) = 0$. $f_1(B(n+1)) = 1$ implies $f_1(E(n+1)) = 1$, since $B(n+1) \leqslant E(n+1)$. In the same way as in the proof of Lemma 1, (1) and (4), we get $E(n+1) \in \mathfrak{M}(n)^c$ and $\| E(n+1)\| = 1$. Since we have $|(f_1 - f_2)(E(n+1))| = 1$, it follows that

$$\|(f_1 - f_2)\,|_{\mathfrak{M}(n)^c}\| \geqslant 1.$$

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